LOCAL-GLOBAL PRINCIPLE FOR CONGRUENCE SUBGROUPS OF CHEVALLEY GROUPS

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ABSTRACT. In this article we prove Suslin's local-global principle for principal congruence subgroups of Chevalley groups. Let $\mathrm{G}(\Phi, \_)$ be a Chevalley–Demazure group scheme with a root system $\Phi \neq A_1$ and $E(\Phi, \_)$ its elementary subgroup. Let R be a ring and I an ideal of R. Assume additionally that R has no residue fields of 2 elements if $\Phi = C_2$ or G_2.

Theorem. *Let $g \in \mathrm{G}(\Phi, R[X], XR[X])$. Suppose that for every maximal ideal $\mathfrak{m}$ of R the image of g under the localization homomorphism at $\mathfrak{m}$ belongs to $\mathrm{E}(\Phi, R_{\mathfrak{m}}[X], IR_{\mathfrak{m}}[X])$. Then, $g \in \mathrm{E}(\Phi, R[X], IR[X])$.*

The theorem is a common generalization of the result of E.Abe for the absolute case ($I = R$) and H.Apte–P.Chattopadhyay–R.Rao for classical groups. It is worth mentioning that for the absolute case the local-global principle was obtained by V.Petrov and A.Stavrova in more general settings of isotropic reductive groups.

INTRODUCTION

In this article we prove Suslin's local-global principle (LGP) for principal congruence subgroups of Chevalley groups. The LGP in context of lower nonstable K-theory was introduced by Suslin [21] and Quillen [16] for solution of Serre's problem on projective modules over polynomial rings. The localization techniques was developed further in works of Bak [3], Taddei [23] and Vaserstein [26, 25] among others. In the last decade a substantial progress in developing and applying localization methods was made by Bak, Hazrat, Vavilov, Zhang Zuhong, and the second named author [19, 4, 9, 11]. This papers utilize Bak's version of localization procedure where a key lemma is continuity of conjugation in s-adic topology (Bak's key lemma).

Originally Suslin proved the LGP for the special linear group. Later it was extended for orthogonal groups by Suslin–Kopejko [22] and for symplectic group by Grunewald–Mennicke–Vaserstein [8]. For all Chevalley groups the LGP was obtained by Abe [1] with an extra condition (condition (P) is used in the proof of Lemma 1.11, although it is missing in the statement). For isotropic reductive groups the LGP was obtained by Petrov and Stavrova [15]. Relative version of LGP in classical groups was worked out by the first-named author in cooperation with Rao and Chattopadhyay [2], where it was shown to be an important tool for study the group structure on orbit sets of unimodular rows. Rao and his group also found several new applications of the LGP [2, 7, 5, 6, 17, 13].

The LGP[1] was first applied for proving K_1-analogue of Serre's conjecture. The history of this statement for classical groups is the same as for LGP itself. For all Chevalley groups a more general statement, $\mathbb{A}^1$-homotopy invariance of nonstable K_1, was recently obtained by Wendt [29], although the proof is incomplete. In more general settings of isotropic reductive groups the homotopy invariance was independently proved by Stavrova [18].

The key place of the proof of the LGP is the dilation principle (Theorem 6.3). It seems that computations performed in section 4 of the current article are absolutely necessary for the proof of the dilation principle. These computations were missing in Abe's paper [1], that is why he did not prove the general case. Almost the same computations are needed for the proof of Bak's key

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[1]In this article we deal with Suslin's version of the LGP, that serves for computing K_1, unlike Quillen's LGP concerned with K_0.

lemma mentioned above. The dilation principle as well as Bak's key lemma implies normality of elementary subgroup almost immediately. In fact, using ideas of preprint [20] by the second-named author, one can prove standard commutator formulas [26], nilpotent structure of K_1 [10], bounded commutator length [19], and multiple commutator formulas [12] without further computations with group elements.

For the latter problem the relative dilation principle obtained in the current article substantially simplifies the proof. Actually, the dilation principle immediately implies Bak's key lemma which was used in [12] to get the result. On the other hand, commutator calculus of [9, 11] provide formulas which are not easy to get from the dilation principle.

Our proof does not depend on other results on the LGP in Chevalley groups. The absolute case does not use normality of elementary subgroup, but for the relative case we apply standard commutator formulas in section 3. We write all detailes in proofs, although some of them are obvious for specialists.

Those who want to focus on the absolute case can set $I = A$ (or $I = R$ in the last section) and skip sections 2 and 3. Moreover, the case $\Phi = C_l$ is proved by Grunewald–Mennicke–Vaserstein [8]. Therefore, section 5 can also be omitted for the absolute case modulo this paper.

1. Notation and preliminaries

All rings in the article are assumed to be commutative with unity and all ring homomorphisms preserve the unity. Let J be an ideal of a ring A. By J^m we denote the product of J with itself m times, i. e. J^m is an additive subgroup generated by all products $r_1 \dots r_m$, where $r_1, \dots, r_m \in J$.

For a multiplicative subset S of A we denote by $S^{-1}A$ the localization of A at S. The localization homomorphism $A \to S^{-1}A$. In the current article we use only two kinds of localizations.

- Principal localizations, where $S = \{s, s^2, \dots\}$ for some element $s \in A$. Principal localizations are denoted by A_s and the corresponding localization homomorphism by λ_s.
- Localizations at a maximal ideal. This means that $S = A \setminus \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A. In this case localization is denoted by $A_\mathfrak{m}$ and the localization homomorphism by $\lambda_\mathfrak{m}$.

Let G be an arbitrary group. We follow standard group-theoretical notation:

- if $a, b \in G$, then $[a, b] = a^{-1}b^{-1}ab$ denotes their commutator whereas $a^b = b^{-1}ab$ stands for the conjugate to a by b;
- if F and H are subsets of G, then $[F, H]$ is the mixed commutator subgroup, i. e. the subgroup generated by commutators $[a, b]$ for all $a \in F$ and $b \in H$;
- double commutators are left-normed, i. e. $[F, H, K] = \big[[F, H], K\big]$, where F, H, K can be understood as subsets or elements of G.

Several times we apply the 3 subgroup lemma which asserts that

$$[F, H, K] \le [K, F, H] \cdot [H, K, F] \tag{1}$$

if F, H, K are normal subgroups of G.

Let $\mathrm{G}(\Phi, \_)$ denotes a Chevalley-Demazure group scheme with a root system Φ. We suppress the weight lattice from the notation. The matter is that we use only standard commutator formulas, which are known for all weight lattices, and elementary computations, i. e. computations in the Steinberg group.

Let A be a ring. *The following condition on Φ and A is used throughout the article.*

Condition 1.1. *$\Phi \ne A_1$. If $\Phi = C_2$ or G_2, then A has no residue fields of 2 elements*

Let I be an ideal of A. The principal congruence subgroup $\mathrm{G}(\Phi, A, I)$ is the kernel of the reduction homomorphism $\mathrm{G}(\Phi, A) \to \mathrm{G}(\Phi, A/I)$ modulo I. By $\mathrm{E}(\Phi, I)$ we denote the subgroup, generated by all root unipotent elements $x_\alpha(r)$ of level I (which means that $r \in I$). If J is another ideal of I, then $\mathrm{E}(\Phi, J, IJ)$ denotes the normal closure of $\mathrm{E}(\Phi, IJ)$ in $\mathrm{E}(\Phi, J)$. In particular, for $J = A$ we obtain the usual definition of the relative elementary subgroup $\mathrm{E}(\Phi, A, I)$ of level I.

The mixed commutator subgroup $[\mathrm{E}(\Phi, A, I), \mathrm{E}(\Phi, A, J)]$ plays an essential role in our considerations. Since this subgroup appears very often, we introduce special notation for it.

$$E_{I,J} = E_{I,J}(\Phi, A) = [\mathrm{E}(\Phi, A, I), \mathrm{E}(\Phi, A, J)] \tag{2}$$

If $J = A$ and condition 1.1 in satisfied, then the (absolute) standard commutator formulas

$$[\mathrm{G}(\Phi, A, I), \mathrm{E}(\Phi, A)] = [\mathrm{E}(\Phi, A, I), \mathrm{G}(\Phi, A)] = [\mathrm{E}(\Phi, A, I), \mathrm{E}(\Phi, A)] = \mathrm{E}(\Phi, A, I) \tag{3}$$

hold. They were obtained by Taddei [23] and Vaserstein [26]. The situation with relative standard commutator formula is more subtle. The inclusion

$$E_{I,J} \geq \mathrm{E}(\Phi, A, IJ) \tag{4}$$

is known to fail for $\Phi = C_l$, if 2 is not invertible in A. This exceptional[2] case is considered in section 5. The case of Sp_{2n} is already done in [2] provided that 2 is invertible in A, although the latter condition is omitted in "Blanket assumptions" of that paper. Without this condition Lemma 3.6 of [2] is probably wrong. Thus, section 5 is to fill this gap.

Actually, result of section 5 hold in general and they are sufficient for the proof of the main theorems. On the other hand, we want to state technical results of sections 2–4 in a sharp form as they are important themselves. Therefore, the following condition is assumed in Lemmas 3.4, 4.1, and 4.2.

Condition 1.2.

- $\Phi \neq A_1$.
- *If $\Phi = G_2$, then A has no residue fields of 2 elements.*
- *If $\Phi = C_l$, then 2 is invertible in A.*

Lemma 1.3 ([11, Lemma 17])**.** *Under condition 1.2 inclusion (4) is valid.*

The reverse inclusion to 4 is valid if $I + J = A$ (see [28, Theorem 5] for the case $\Phi = A_l$), in general a counterexample was constructed in [14]. On the other hand, under condition 1.1 the formula

$$[\mathrm{G}(\Phi, A, I), \mathrm{E}(\Phi, A, J)] = [\mathrm{E}(\Phi, A, I), \mathrm{E}(\Phi, A, J)] \tag{5}$$

follows from 3 subgroup lemma (formula (1)) and absolute standard commutator formulas (3) almost immediately (see the proof of [28, Theorem 2]). We reproduce the proof in section 3 for completeness.

To prove the relative dilation principle and the relative LGP we shall use only absolute standard commutator formulas (3).

We call J a splitting ideal if $A = R \oplus J$ as additive groups, where R is a subring of A. Of course, in this case $R \cong A/J$. Equivalently, J is a splitting ideal iff it is a kernel of a retraction $A \to R \subseteq A$. In the current article we use only one example of splitting ideals. Namely, if $A = R[t]$ is a polynomial ring, then the ideal tA is obviously splitting.

The following statement is called (absolute) splitting principle. It follows easily from Lemma 2.1 (which itself is very simple). It was formulated in [1, Proposition 1.6] and [2, Lemma 3.3].

Lemma 1.4. *If J is a splitting ideal of a ring A, then* $\mathrm{E}(\Phi, A) \cap \mathrm{G}(\Phi, A, J) = \mathrm{E}(\Phi, A, J)$.

2. Relative splitting principle

Let J be a splitting ideal of A. For relative groups the absolute splitting principle 1.4 implies that

$$\mathrm{E}(\Phi, A, I) \cap \mathrm{G}(\Phi, A, J) = \mathrm{E}(\Phi, A, I) \cap \mathrm{E}(\Phi, A, J).$$

To prove the relative dilation principle (Theorem 6.3) we need a smaller group on the right hand side of the formula above. To do this we must assume that I is extended from an ideal of A/J, i. e. $I = I'A$ for some ideal I' of the image of A/J.

The splitting principle bases on the following simple group-theoretical lemma. The proof is an easy group theoretical exercise and will be left to the reader.

[2] Exceptional groups are not exceptions in this settings, the only exception is the classical group Sp_{2n}.

Lemma 2.1. *Let $G = G' \ltimes G''$ be a semidirect product (so that G'' is a normal subgroup). Let $H' \leq G'$ and $H'' \leq G''$ be such that $H = H'H''$ is a subgroup (this amounts to say that H' normalizes H''). Then $H'' = H \cap G''$.*

To state the relative splitting principle in a form independent of condition 1.2 we introduce the following notation: $\bar{E}_{I,J} = E_{I,J}\,\mathrm{E}(\Phi, A, IJ)$. By Lemma 1.3, under condition 1.2 we have $\bar{E}_{I,J} = E_{I,J}$.

Lemma 2.2 (Relative Splitting Principle)**.** *Suppose that J is a splitting ideal of A so that $A = R \oplus J$ for some subring R of A. Let I' be an ideal of R. Put $I = AI'$. Then*

$$\mathrm{E}(\Phi, A, I) \cap \mathrm{G}(\Phi, A, J) = \bar{E}_{I,J}.$$

Proof. Applying the functor $\mathrm{G}(\Phi, \_)$ to $R \hookrightarrow A \twoheadrightarrow R$ we get a retraction $\mathrm{G}(\Phi, R) \hookrightarrow \mathrm{G}(\Phi, A) \twoheadrightarrow \mathrm{G}(\Phi, R)$ and the kernel of the latter map is $\mathrm{G}(\Phi, R, J)$. Therefore, $\mathrm{G}(\Phi, A) = \mathrm{G}(\Phi, R) \ltimes \mathrm{G}(\Phi, A, J)$. Clearly, $\mathrm{E}(\Phi, R, I') \leq \mathrm{G}(\Phi, R)$, $\bar{E}_{I,J} \leq \mathrm{G}(\Phi, R, J)$, and $E_{I,J}$ is normal in $\mathrm{E}(\Phi, A)$ (by definition of relative elementary subgroup). By Lemma 2.1 it suffices to show that

$$\mathrm{E}(\Phi, A, I) = \bar{E}_{I,J}\,\mathrm{E}(\Phi, R, I').$$

Since $\mathrm{E}(\Phi, A, I)$ is normal in $\mathrm{E}(\Phi, A)$, the right hand side is contained in the left hand side.

Note that ideal I is additively generated by the elements rs, where $r \in A$ and $s \in I'$. Write $r = r' + r''$ for some $r' \in R$ and $r'' \in J$. Then $rs = r's + r''s \in I' + IJ$. It follows that $I = I' \oplus IJ$.

The group $\mathrm{E}(\Phi, A, I)$ is generated by the elements $x_\alpha(u + v)^b$, where $\alpha \in \Phi$, $u \in I'$, $v \in IJ$, and $b \in \mathrm{E}(\Phi, A)$. Since $x_\alpha(v)^b \in \mathrm{E}(\Phi, A, IJ) \leq \bar{E}_{I,J}$, it remains to show that $x_\alpha(u)^b \in E_{I,J}\,\mathrm{E}(\Phi, R, I')$. By the absolute splitting principle we can write $b = cd$, where $c \in \mathrm{E}(\Phi, R)$ and $d \in \mathrm{E}(\Phi, A, J)$. Then $g = x_\alpha(u)^c \in \mathrm{E}(\Phi, R, I')$ and $x_\alpha(u)^b = g^d = [d, g^{-1}]g \in E_{I,J}\,\mathrm{E}(\Phi, R, I')$. □

3. Commutator formulas

The only commutator calculus with particular elements are used in sections 4 and 5. In the current section we perform some commutator calculus with subgroups in spirit of [28].

Let H be an arbitrary algebraic group scheme over a ring Z and A a Z-algebra (for a Chevalley group scheme one can take $Z = \mathbb{Z}$, the notation is to illustrate this). Denote by $H(A, I)$ the kernel of the reduction homomorphism $H(A) \to H(A/I)$. In the following lemma we use a faithful representation of the group scheme H.[3] For $H = \mathrm{GL}_n$ the lemma was obtained in [27]. Throughout this section I and J denote arbitrary ideals of a ring A.

Lemma 3.1. $[H(A, I), H(A, J)] \leq H(A, IJ)$.

Proof. Let $H \to \mathrm{GL}_n$ be a faithful rational representation of H. Identifying elements of $H(A)$ with their images in $\mathrm{GL}_n(A)$ we have $H(A, I) = H(A) \cap \mathrm{GL}_n(A, I)$. By [27, Lemma 3] the statement holds for $H = \mathrm{GL}_n$. Thus,

$$[H(A, I), H(A, J)] \leq H(A) \cap [\mathrm{GL}_n(A, I), \mathrm{GL}_n(A, J)] \leq H(A) \cap \mathrm{GL}_n(A, IJ) = H(A, IJ).$$

□

Now we can prove equation 5. The proof is the same as for [28, Teorem 2].

Lemma 3.2. *Under condition* 1.1 *we have*

$$E_{I,J} \leq [\mathrm{G}(\Phi, A, I), \mathrm{E}(\Phi, A, J)] \leq \bar{E}_{I,J}.$$

Under condition 1.2 *all inequalities turn into equalities.*

[3] A faithful representation is just a special choice of generators of the affine algebra of H. In fact one can take any set of generators to prove the lemma, but arguments with use of representation are more intuitively clear.

Proof. The former inequality is trivial. By condition 1.1 we have absolute standard commutator formulas (3). Therefore,

$$\begin{aligned}
&[\mathrm{G}(\Phi,A,I),\mathrm{E}(\Phi,A,J)] = \big[\mathrm{G}(\Phi,A,I),[\mathrm{E}(\Phi,A,J),\mathrm{E}(\Phi,A)]\big] \le \qquad \text{(by 3 subgroup lemma)}\\
&[\mathrm{G}(\Phi,A,I),\mathrm{E}(\Phi,A,J),\mathrm{E}(\Phi,A)]\cdot[\mathrm{E}(\Phi,A),\mathrm{G}(\Phi,A,I),\mathrm{E}(\Phi,A,J)] \le \qquad \text{(by Lemma 3.1)}\\
&[\mathrm{G}(\Phi,A,IJ),\mathrm{E}(\Phi,A)]\cdot[\mathrm{E}(\Phi,A,I),\mathrm{E}(\Phi,A,J)] = \mathrm{E}(\Phi,A,IJ)E_{I,J}
\end{aligned}$$

□

Let I,J,K be ideals of a commutative ring A. The following useful fact was observed for GL_n in [12].

Lemma 3.3. *Under condition* 1.1 *we have*

$$E_{IJ,K} \le [\bar{E}_{I,J},\mathrm{E}(\Phi,A,K)] \le \bar{E}_{IJ,K}.$$

Under condition 1.2 *all inequalities turn into equalities.*

Proof. Since all the subgroups involved are normal, we have $[\bar{E}_{I,J},\mathrm{E}(\Phi,A,K)] = [E_{I,J},\mathrm{E}(\Phi,A,K)]\cdot E_{IJ,K}$. Therefore, it remains to show that $[E_{I,J},\mathrm{E}(\Phi,A,K)] \le \bar{E}_{IJ,K}$, but this follows immediately from lemmas 3.1 and 3.2. □

From now on we assume that condition 1.2 holds until the middle of section 4.

The following statement is the first ingredient for the dilation principle. In the proof of the dilation principle it is used with $J = At$, where $A = R[t]$ is a polynomial ring. The lemma allows to move an independent variable t from the first subgroup of the mixed commutator to the second one.

Lemma 3.4. *Let I,J,K be ideals of A. Under condition* 1.2 *we have* $E_{IJ^2,K} \le \mathrm{E}(\Phi,IJK)E_{IJ,KJ}$.

Proof. By Lemma 1.3 for any pair L,M of ideals we have $\mathrm{E}(\Phi,A,LM) \le [\mathrm{E}(\Phi,A,L),\mathrm{E}(\Phi,A,M)]$. Therefore,

$$\begin{aligned}
&E_{IJ^2,K} = [\mathrm{E}(\Phi,A,IJ^2),\mathrm{E}(\Phi,A,K)] \le [\mathrm{E}(\Phi,A,IJ),\mathrm{E}(\Phi,A,J),\mathrm{E}(\Phi,A,K)] \le\\
&\qquad\qquad \text{(by the 3 subgroups lemma)}\\
&[\mathrm{E}(\Phi,A,K),\mathrm{E}(\Phi,A,IJ),\mathrm{E}(\Phi,A,J)]\cdot[\mathrm{E}(\Phi,A,J),\mathrm{E}(\Phi,A,K),\mathrm{E}(\Phi,A,IJ)] \le\\
&\qquad\qquad \text{(by Lemma 3.3)}\\
&[\mathrm{E}(\Phi,A,KIJ),\mathrm{E}(\Phi,A,J)]\cdot[\mathrm{E}(\Phi,A,KJ),\mathrm{E}(\Phi,A,IJ)] \le \mathrm{E}(\Phi,IJK)[\mathrm{E}(\Phi,A,IJ),\mathrm{E}(\Phi,A,KJ)].
\end{aligned}$$

□

4. Commutator calculus

The following lemma is the second ingredient of the dilation principle. For the absolute case it was formulated (without proof) by J.Tits in [24].

Lemma 4.1. *Under condition* 1.2 *we have* $\mathrm{E}(\Phi,A,IJ^2) \le \mathrm{E}(\Phi,J,IJ)$.

The proof of the above statement bases on the following three lemmas. Let $\alpha\in\Phi$. Denote by $Y(\alpha,J,I)$ the subgroup of $\mathrm{E}(\Phi,A)$ generated by elements $x_\beta(-r)x_\gamma(s)x_\beta(r)$ for all $\beta,\gamma\in\Phi\setminus\{\pm\alpha\}$, $r\in J$ and $s\in I$. The next statement shows that we can decompose an elementary root unipotent element of level IJ^2 to a product of elementary unipotents, avoiding a given root and its inverse (the case $\Phi = C_l$ and 2 is not invertible in A is exceptional as above).

Lemma 4.2. *Let $\alpha\in\Phi$. If condition 1.2 holds, then $x_\alpha(IJ^2) \le Y(\alpha,J,IJ)$.*

Proof. Take $\beta,\gamma\in\Phi$ such that $\alpha=\beta+\gamma$. For any $p\in I$ and $q,r\in J$ write the Chevalley commutator formula

$$[x_\beta(pq),x_\gamma(\varepsilon r)] = x_\alpha(N_{\beta\gamma 11}\varepsilon pqr)\prod x_{i\beta+j\gamma}(N_{\beta\gamma ij}\varepsilon p^i q^i r^j),$$

where ε is an invertible element of A and the product is taken over all $i,j>0$ such that $i\beta+j\gamma\in\Phi$ and $i+j>2$. Clearly, β, γ and $i\beta+j\gamma$ with $i+j>2$ cannot be equal to $\pm\alpha$. Therefore, the left

hand side and each factor of the product belong to $Y(\alpha, J, IJ)$. Hence, it suffices to find β and γ such that $N_{\beta\gamma 11}$ is invertible in A and put $\varepsilon = N_{\beta\gamma 11}^{-1}$.

If α is a short root, then it is a sum of a short and long root. (if all roots have the same length, we call them long). In this case α, β and γ span the root system of type C_2 or G_2 and $N_{\beta\gamma 11} = \pm 1$.

If α, β and γ are long roots, then $N_{\beta\gamma 11} = \pm 1$. Such a decomposition for a long root is available in all root systems except $\Phi = C_l$. By condition 1.2, if $\Phi = C_l$, then 2 is invertible in A. Any long root $\alpha \in C_l$ is a sum of two short roots β and γ. Now, the formula $[x_\beta(pq), x_\gamma(r/2)] = x_\alpha(pqr)$ completes the proof. □

The rest of the section does not depend on condition 1.2.

Lemma 4.3 (Vaserstein [26, Theorem 2])**.** *Under condition 1.1 the group* $\mathrm{E}(\Phi, A, I)$ *is generated by the elements* $x_{-\alpha}(-r)x_\alpha(s)x_{-\alpha}(r)$*, for all* $\alpha \in \Phi$*,* $r \in A$ *and* $s \in I$*.*

The next lemma follows easily from the Chevalley commutator formula.

Lemma 4.4. *For any* $\beta \neq \alpha \in \Phi$*,* $r \in A$ *and* $s \in I$ *we have*

$$x_{-\alpha}(-r)x_\beta(s)x_{-\alpha}(r) \in \mathrm{E}(\Phi, I).$$

It follows that $x_{-\alpha}(-r)Y(\alpha, J, IJ)x_{-\alpha}(r) \subseteq \mathrm{E}(\Phi, J, IJ)$*.*

Proof of Lemma 4.1*.* By Lemma 4.3 it suffices to prove that $x_{-\alpha}(-r)x_\alpha(s)x_{-\alpha}(r) \in \mathrm{E}(\Phi, J, IJ)$ for all $\alpha \in \Phi$, $s \in IJ^2$, and $r \in A$. By Lemma 4.2 $x_\alpha(s) \in Y(\alpha, J, IJ)$. Now, the result follows from Lemma 4.4. □

5. Symplectic group

In this section we make necessary changes in Lemmas 1.3, 3.4, 4.2, and 4.1 for the exceptional case $\Phi = C_l$. With this end we need to distinguish between J^m and $J^{\boxed{m}}$, which denotes the ideal generated by r^m for all $r \in J$. The following lemma is commonly known. We give a one row proof for completeness.

Lemma 5.1. *Suppose that a ring* A *has no residue fields of* 2 *elements. Then ideal generated by* $r^2 - r$ *for all* $r \in A$ *coincides with the whole ring.*

Proof. Suppose that $r^2 - r$ belongs to a maximal ideal $\mathfrak{m}$ for all $r \in A$. Then $A/\mathfrak{m}$ is a field consisting of idempotents. Hence, $A/\mathfrak{m} = \mathbb{F}_2$, a contradiction. □

The following statement substitutes Lemma 1.3. Idea of the proof is borrowed from [11].

Lemma 5.2. *Under condition 1.1 we have* $\mathrm{E}(\Phi, A, I^{\boxed{2}}J + 2IJ + IJ^{\boxed{2}}) \leq [\mathrm{E}(\Phi, A, I), \mathrm{E}(\Phi, A, J)]$.

Proof. In view of Lemma 1.3 it suffices to consider the case $\Phi = C_l$. Denote by H the right hand side of the inclusion. Since H is normal in $\mathrm{E}(C_l, A)$, it remains to show that $x_\alpha(r)$ belongs to the right hand side for all $\alpha \in C_l$ and $r \in I^{\boxed{2}}J + 2IJ + IJ^{\boxed{2}}$.

If $l \geq 3$, then any short root lies in a subsystem of roots of type A_2. For A_2 we have even stronger inclusion (4).

Now, let α be a long root. Then there exists a long root β and a short root γ such that $\alpha = \beta + 2\gamma$. Note that $\beta + \gamma$ is a short root. By the formula $x_\alpha(2pq) = [x_{\beta+\gamma}(\pm p), x_\gamma(\pm q)]$ with $p \in I$ and $q \in J$ we get $x_\alpha(2IJ) \leq H$. Then, apply the relation $x_\alpha(pq^2) = [x_\beta(\pm p), x_\gamma(\pm q)]x_{\beta+\gamma}(\pm pq)$. By the first paragraph of the proof $x_{\beta+\gamma}(\pm pq) \in H$. Clearly, the first factor of the right hand side lies in H provided that $p \in I$, $q \in J$ or $q \in I$, $p \in J$. This shows that $x_\alpha(I^{\boxed{2}}J) \leq H$ and $x_\alpha(IJ^{\boxed{2}}) \leq H$.

Now, let $l = 2$, α and β are long root, γ is a short root, and $\alpha = \beta + 2\gamma$. Take $p \in I$, $q \in J$ and $r, s \in A$. Write Chevalley commutator formulas

$$[x_\beta(\pm sp), x_\gamma(\pm rq)] = x_{\beta+\gamma}(\pm rpq)x_\alpha(sr^2pq^2) \in H \text{ and}$$
$$[x_\beta(\pm srp), x_\gamma(\pm q)] = x_{\beta+\gamma}(\pm rpq)x_\alpha(srpq^2) \in H.$$

The difference between right hand sides of these formulas is equal to $x_\alpha\bigl(s(r^2-r)pq^2\bigr)$ and lies in H for all $r \in R$. By Lemma 5.1 $x_\alpha(pq^2) \in H$. Similarly, $x_\alpha(qp^2) \in H$. The proof of inclusion $x_\alpha(2IJ) \leq H$ is the same as for $l \geq 3$.

Finally, displayed formulas above with $r = s = 1$ shows that $x_{\beta+\gamma}(pq) \in H$ as the last factor is in H. □

The next result is to substitute Lemma 3.4. The proof is essentially the same.

Lemma 5.3. *Let I, J, K be ideals of A. Then $\bar{E}_{IJJ\boxed{2},K} \leq \mathrm{E}(\Phi, A, IJK) \cdot E_{IJ,KJ}$.*

Proof. Clearly, $\mathrm{E}(\Phi, A, IJJ\boxed{2}K) \leq \mathrm{E}(\Phi, A, IJK)$. By Lemma 5.2 we have $\mathrm{E}(\Phi, A, IJJ\boxed{2}) \leq E_{IJ,J}$. Now the proof almost coincides with the proof of Lemma 3.4. Namely, by the 3 subgroup lemma we have $[E_{IJ,J}, \mathrm{E}(\Phi, A, K)] \leq [E_{IJ,K}, \mathrm{E}(\Phi, A, J)] \cdot [E_{J,K}, \mathrm{E}(\Phi, A, IJ)]$. Now, by Lemma 3.2 the right hand side of the above formula is contained in $\bar{E}_{IJK,J}\bar{E}_{JK,IJ} \leq \mathrm{E}(\Phi, A, IJK)E_{JK,IJ}$. □

Now we take care about Lemma 4.1. First we prove a counterpart of Lemma 4.2.

Lemma 5.4. *Let $\alpha \in \Phi$. Then $x_\alpha(IJJ\boxed{2}) \leq Y(\alpha, J, IJ)$.*

Proof. In view of Lemma 4.2 it suffices to consider the case $\Phi = C_l$. The proof is the same as for the previous lemma replacing I by IJ. Details are left to the reader. □

Lemma 5.5. $\mathrm{E}(\Phi, A, IJJ\boxed{2}) \leq \mathrm{E}(\Phi, J, IJ)$.

Proof. By Lemma 4.3 it suffices to prove that $x_{-\alpha}(-r)x_\alpha(s)x_{-\alpha}(r) \in \mathrm{E}(\Phi, J, IJ)$ for all $\alpha \in \Phi$, $s \in IJJ\boxed{2}$ and $r \in A$. By Lemma 5.4 $x_\alpha(s) \in Y(\alpha, J, IJ)$. Now, the result follows from Lemma 4.4. □

6. Relative dilation principle

The next statement follows immediately from Lemmas 5.3 and 5.5.

Corollary 6.1. *Let K be an ideal in A. Under condition 1.1 we have $\bar{E}_{K,At^9} \leq \mathrm{E}(\Phi, At, Kt)$.*

Proof. Lemma 5.3 with $I = A$ and $J = At^3$ shows that $\bar{E}_{K,At^9}$ is contained in $\mathrm{E}(\Phi, A, Kt^3)E_{At^3,Kt^3} \leq \mathrm{E}(\Phi, A, Kt^3)$. The latter group is contained in $\mathrm{E}(\Phi, At, Kt)$ by Lemma 5.5. □

Let R be a ring and $s \in R$. Using relative splitting principle and previous corollary it is easy to see that if $g(t) \in \mathrm{E}(\Phi, R_s[t], IR_s[t]) \cap \mathrm{G}(\Phi, R_s[t], tR_s[t])$, then $g(s^n t)$ lies in the image of $\mathrm{E}(\Phi, R[t], IR[t])$ under the localization homomorphism for some $n \in \mathbb{N}$. But localization homomorphism is not injective in general. The following lemma is to get around this problem. For the general linear group this trick was noticed in [21]. In [15, Lemma 14] this is stated for all affine schemes. However, we believe that our detailed proof may be usefull for nonspecialists. Recall that λ_s denotes the principal localization homomorphism $R \to R_s$, where $s \in R$.

Lemma 6.2. *Let H be an algebraic group scheme. Let $g(t), h(t) \in H(R[t], tR[t])$ and $s \in R$ be such that $\lambda_s(g) = \lambda_s(h)$. Then there exists $m \in \mathbb{N}$ such that $g(s^m t) = h(s^m t)$.*

Proof. As in Lemma 3.1 we consider a faithful representation of the group scheme H and identify elements of $H(A)$ with their images in $\mathrm{GL}_n(A)$ for any ring A. Recall that a_{ij} denotes the entry of a matrix a in position (i, j). Note that $\lambda_s(g) = \lambda_s(h)$ implies that there exists $m \in \mathbb{N}$ such that $s^m g_{ij} = s^m h_{ij}$ for all $i, j = 1, \ldots, n$. Since $g, h \in H(A[t], tA[t]) \subseteq \mathrm{GL}_n(A[t], tA[t])$, we have $g_{ij} - \delta_{ij} = t\tilde{g}_{ij}$ and $h_{ij} - \delta_{ij} = t\tilde{h}_{ij}$ for some $\tilde{g}_{ij}, \tilde{h}_{ij} \in R[t]$. Since t is not a zero divisor, $s^m\tilde{g}_{ij} = s^m\tilde{h}_{ij}$ for all $i, j = 1, \ldots, n$ which implies the result. □

Theorem 6.3 (Relative Dilation Principle)**.** *Let $g(t) \in \mathrm{G}(\Phi, R[t], tR[t])$ be such that $\lambda_s(g) \in \mathrm{E}(\Phi, R_s[t], IR_s[t])$. Then there exists $l \in \mathbb{N}$ such that $g(s^l t) \in \mathrm{E}(\Phi, R, IR[t])$.*

Proof. Let t' be another independent variable and $R' = R_s[t,t']$. Put $f(t,t') = \lambda_s\big(g(tt')\big)$. Then $f(t,t') \in \mathrm{E}(\Phi, R', IR') \cap \mathrm{G}(\Phi, R', tR')$. By Lemma 2.2 $f(t,t') \in \bar{E}_{IR',t'R'}(\Phi, R')$ and by Corollary 6.1 $f(t,t'^9) \in \mathrm{E}(\Phi, t'R', t'IR')$. By the definition of the latter group,

$$f(t,t'^9) = \prod_j x_{\alpha_j}\left(\frac{t'v_j}{s^{k_j}}\right)^{a_j}, \text{ where } a_j = \prod_i x_{\alpha_{ji}}\left(\frac{t'u_{ji}}{s^{k_{ji}}}\right),$$

$k_j, k_{ji} \in \mathbb{N}$, $u_{ji} \in R[t,t']$, $v_j \in IR[t,t']$ and $\alpha_j, \alpha_{ji} \in \Phi$. Clearly, if $m \geq k_j, k_{ji}$ for all j, i, then

$$f\big(t, (s^m t')^9\big) \in \lambda_s\big(\mathrm{E}(\Phi, R[t], It'R[t,t'])\big).$$

Take $n \geq 9\max(k_j, k_{ij})$ and put $t' = 1$ to get $\lambda_s\big(g(ts^n)\big) = f(t, s^n) \in \lambda_s\big(h(t)\big)$ for some $h(t) \in \mathrm{E}(\Phi, R[t], I[t])$. By Lemma 6.2(ii) there exists $k \in \mathbb{N}$ such that $g\big(ts^k s^n\big) = h(ts^k) \in \mathrm{E}(\Phi, R[t], IR[t])$, which completes the proof. □

7. Relative Local-Global principle

The proof of the LGP follows the same ideas as original Suslin's proof. However, the exposition is different. Namely, we do not use induction but introduce a "generic ring" for the problem. For us this arguments seem to be a little bit easier, but this is a matter of taste.

Recall that $\lambda_{\mathfrak{m}}$ denotes the localization homomorphism at a maximal ideal $\mathfrak{m}$.

Theorem 7.1. *Let I be an ideal of a commutative ring R and $g = g(X) \in \mathrm{G}(\Phi, R[X], XR[X])$. Suppose that $\lambda_{\mathfrak{m}}(g) \in \mathrm{E}(\Phi, R_{\mathfrak{m}}[X], IR_{\mathfrak{m}}[X])$ for every maximal ideal $\mathfrak{m}$ of R.*

Then, $g \in \mathrm{E}(\Phi, R[X], IR[X])$.

Proof. Denote by S the set of all $s \in R$ such that $\lambda_s(g) \in \mathrm{E}(\Phi, R_s[X], IR_s[X])$. We show that S is unimodular. Indeed, given a maximal ideal $\mathfrak{m}$ of R we can write $\lambda_{\mathfrak{m}}(g)$ as a product

$$\lambda_{\mathfrak{m}}(g) = \prod_j x_{\alpha_j}\left(\frac{v_j}{r_j}\right)^{a_j}, \text{ where } a_j = \prod_i x_{\alpha_{ji}}\left(\frac{u_{ji}}{r_{ji}}\right),$$

$r_j, r_{ji} \in R \setminus \mathfrak{m}$, $u_{ji} \in R[t,t']$, $v_j \in IR[X]$ and $\alpha_j, \alpha_{ji} \in \Phi$.

Multiplying r_j and r_{ij} over all i, j we get an element $s \in R \setminus \mathfrak{m}$. It is easy to see that $s \in S$. Therefore, S is not contained in any maximal ideal of R, hence it is unimodular. Now, 1 is a linear combination of a finite subset of elements $s_1, \dots, s_m \in S$ with coefficients from R. Thus, we obtain a unimodular sequence $s_1, \dots, s_m$ such that

$$\lambda_{s_k}(g) \in \mathrm{E}(\Phi, A_{s_k}[X], IA_{s_k}[X]) \text{ for all } k = 1, \dots, m.$$

Consider the ring $A = R[X, t_1, \dots, t_m]/(t_1 + \dots + t_m = 1)$. Since $R[X]$ is a subring of A, we can view g as an element of $\mathrm{G}(\Phi, A, XA)$. Put

$$g_1 = g\Big(Xt_1\Big) \text{ and } g_k = g\Big(X\sum_{i=1}^{k-1} t_i\Big)^{-1} g\Big(X\sum_{i=1}^{k} t_i\Big) \text{ for } k = 2, \dots, m.$$

Then $g_k \in \mathrm{G}(\Phi, A, t_kA)$ for all k and $g = g_1 \dots g_m$. Fix $k \in \{1, \dots, m\}$. Note that $A = R^{(k)}[t_k]$ is a polynomial ring, where $R^{(k)} = R[X, t_i \mid i \neq k, j]$ and j is some index distinct from k (in this case we view t_j as an abbreviation of $1 - \sum_{i \neq j} t_i$). Since for any $s \in R$ localization homomorphism λ_s commutes with evaluation homomorphisms $X \mapsto X\sum_{i=1}^{k} t_i$, we have $\lambda_{s_k}(g_k) \in \mathrm{E}(\Phi, R^{(k)}_{s_k}[t_k], IR^{(k)}_{s_k}[t_k])$. By Theorem 6.3 with $R = R^{(k)}$ and $t = t_k$, there exists $l_k \in \mathbb{N}$ such that $g_k(s_k^{l_k} t_k) \in \mathrm{E}(\Phi, R^{(k)}[t_k], IR^{(k)}[t_k])$.

The sequence $s_1^{l_1}, \dots, s_m^{l_m}$ is a unimodular sequence of elements of R (otherwise all $s_i^{l_i}$ would lie in the same maximal ideal $\mathfrak{m}$ which would imply that all s_i's are in $\mathfrak{m}$, a contradiction). Fix $p_1, \dots, p_m \in R$ such that $1 = \sum_{k=1}^{m} s_k^{l_k} p_k$. Denote by $\varepsilon_k : A \to R^{(k)}$ the evaluation homomorphism sending t_k to $s_k^{l_k} p_k$. Then $\varepsilon_k(g_k) \in \mathrm{E}(\Phi, R^{(k)}, IR^{(k)})$.

Now, let $\varepsilon : A \to R[X]$ be the homomorphism identical on $R[X]$ sending t_k to $s_k^{l_k} p_k$ for all $k = 1, \dots, m$ (by the choice of $p_1, \dots, p_m$ such a homomorphism exists). Since ε factors through ε_k for each k, we have $\varepsilon(g_k) \in \mathrm{E}(\Phi, A[X], IA[X])$, hence $g = \varepsilon(g) \in \mathrm{E}(\Phi, A[X], I[X])$, which completes the proof. $\square$

References

1. E. Abe, *Whitehead groups of Chevalley groups over polynomial rings*, Comm. Algebra **11** (1983), no. 12, 1271–1307.
2. H. Apte, P. Chattopadhyay, and R. Rao, *A local global theorem for extended ideals*, J. Ramanujan Mathematical Society **27** (2012), no. 1, 17–30.
3. A. Bak, *Nonabelian K-theory: The nilpotent class of* K_1 *and general stability*, K-theory **4** (1991), 363–397.
4. A. Bak, R. Hazrat, and N. Vavilov, *Localization-completion strikes again: relative K_1 is nilpotent*, J. Pure and Applied Algebra **213** (2009), 1075–1085.
5. R. Basu, P. Chattopadhyay, and R. Rao, *Some remarks on symplectic injective stability*, Proc. Amer. Math. Soc. **139** (2011), no. 7, 2317–2325.
6. R. Basu and R. Rao, *Injective stability for K_1 of classical modules*, J. Algebra **323** (2010), no. 4, 867–877.
7. P. Chattopadhyay and R. Rao, *Elementary symplectic orbits and improved K_1-stability*, J. K-Theory **7** (2011), no. 2, 389–403.
8. F. Grunewald, J. Mennicke, and L. Vaserstein, *On symplectic groups over polynomial rings*, Math. Zeitschrift **206** (1991), 35–56.
9. R. Hazrat, A. Stepanov, N. Vavilov, and Z. Zhang, *The yoga of commutators*, J. Math. Sci (New-York) **179** (2011), no. 6, 662–678.
10. R. Hazrat and N. Vavilov, *K_1 of Chevalley groups are nilpotent*, J. Pure and Appl. Algebra **179** (2003), 99–116.
11. R. Hazrat, N. Vavilov, and Z. Zhang, *Relative commutator calculus in Chevalley groups*, J.Algebra (2012), to appear.
12. R. Hazrat and Z. Zhang, *Multiple commutator formulas*, Israel J. Math. (2012), to appear.
13. S. Jose and R. Rao, *A local global principle for the elementary unimodular vector group*, Contemp. Math. **390** (2005), 119–125.
14. A. W. Mason, *On subgroups of* $\mathrm{GL}(n, A)$ *which are generated by commutators. II*, J. reine angew. Math. **322** (1981), 118–135.
15. V. Petrov and A. Stavrova, *Elementary subgroups in isotropic reductive groups*, St. Petersburg Mathematical Journal **20** (2009), no. 4, 625–644.
16. D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.
17. R. Rao, R. Basu, and S. Jose, *Injective stability for K_1 of the orthogonal group*, J. Algebra **323** (2010), no. 2, 393–396.
18. A. Stavrova, *Homotopy invariance of non-stable K_1-functors*, Preprint: `http://ArXiv.org/abs/1111.4664`, 2012.
19. A. Stepanov and N. Vavilov, *Length of commutators in Chevalley groups*, Israel J. Math. **185** (2011), 253–276.
20. A. V. Stepanov, *Universal localization for Chevalley groups over rings*, J. Algebra **362** (2012), 12–29.
21. A. A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR. Izvestija **11** (1977), 221–238.
22. A. A. Suslin and V. I. Kopejko, *Quadratic modules and orthogonal group over polynomial rings*, J. Sov. Math. **20** (1982), no. 6, 2665–2691.
23. G. Taddei, *Normalité des groupes élémentaire dans les groupes de Chevalley sur un anneau*, Contemp. Math. **55** (1986), 693–710.
24. J. Tits, *Systemes générateurs de groupes de congruences*, C. R. Acad. Sci., Paris, Sér. A **283** (1976), 693–695.
25. L. N. Vaserstein, *On the normal subgroups of GL_n over a ring*, Lecture Notes in Math. **854** (1981), 456–465.
26. ______, *On normal subgroups of Chevalley groups over commutative rings*, Tôhoku Math. J. **38** (1986), 219–230.
27. N. A. Vavilov and A. V. Stepanov, *Standard commutator formulae*, Vestnik St.Petersburg Univ., Math. (2008), no. 1, 9–14.
28. ______, *Standard commutator formulae, revisited*, Vestnik St.Petersburg Univ., Math. **43** (2010), no. 1, 12–17.
29. Wendt, M., *$\mathbb{A}^1$-homotopy of Chevalley groups*, J. K-Theory **5** (2010), 245–287.

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